Nonrelativistic Factorizable Scattering Theory of Multicomponent Calogero-Sutherland Model

Changrim Ahn, Kong-Ju-Bock Lee

Department of Physics, Ewha Womans University, Seoul, 120-750, Korea

Soonkeon Nam

Department of Physics, Kyung Hee University, Seoul, 130-701, Korea

We relate two integrable models in (1+1) dimensions, namely, multicomponent Calogero-Sutherland model with particles and antiparticles interacting via the hyperbolic potential and the nonrelativistic factorizable S-matrix theory with $SU(N)$-invariance. We find complete solutions of the Yang-Baxter equations without implementing the crossing symmetry, and one of them is identified with the scattering amplitudes derived from the Schrödinger equation of the Calogero-Sutherland model. This particular solution is of interest in that it cannot be obtained as a nonrelativistic limit of any known relativistic solutions of the $SU(N)$-invariant Yang-Baxter equations.

Recently, there has been a great interest in integrable quantum systems with long-range interactions. Of these the Calogero-Sutherland model (CSM) [1,2], which is an n-body non-relativistic quantum mechanical system with long-range two-body potentials, has been considered in connection with integrable spin-chains with long-range interactions [3,4], random matrix theory [5], and fractional statistics [6]. Furthermore it was shown in [7], the S-matrix of the model with hyperbolic potentials is the same as the non-relativistic limit of the sine-Gordon soliton S-matrix. That is, the S-matrix of CSM is a $O(2)$-symmetric solution of the Yang-Baxter equations, together with unitarity condition, but without the crossing symmetry. This connection was also observed for the boundary sine-Gordon equation in its relation to BC_n type CSM [8].

In this letter, we establish a connection between the multicomponent CSM, the n-body quantum mechanical system of colored particles and antiparticles interacting via integrable long-range potential of hyperbolic-type and nonrelativistic factorizable S-matrix theory with $SU(N)$-invariance. First we derive the scattering amplitudes from the eigenfunctions of the CSM Hamiltonian. To relate this to the S-matrix theory, we obtain complete solutions of the $SU(N)$-invariant Yang-Baxter equations. In relativistic scattering theory, this model has been solved completely in [9]. Without implementing the crossing symmetry we find a new class of solutions and, interestingly, one of these corresponds to the scattering amplitudes of the CSM. This means that the multicomponent CSM cannot be obtained as a nonrelativistic limit of any relativistic systems, contrary to the the sine-Gordon model ($O(2)$ case) mentioned above.

We begin with the multicomponent CSM where the Hamiltonian is given by

$$H = -\sum_i^n \frac{\partial^2}{\partial x_i} + \sum_{i<j} \frac{\lambda(\lambda + P_{ij})}{\sinh^2(x_i - x_j)} \tag{1}$$

where P_{ij} is the exchange operator of the colors of (anti)particles at x_i and x_j. This model has been shown to be integrable by several authors [10,11]. Since $P_{ij}^2 = I$, we can define the eigenstates of P_{ij} as $|\pm\rangle = \frac{1}{\sqrt{2}}(|\sigma_i \sigma_j\rangle \pm |\sigma_j \sigma_i\rangle)$ such that $P_{ij}|\pm\rangle = \pm|\pm\rangle$.

To obtain the two-body S-matrix, we consider the scattering eigenstates of the Schrödinger equation $\left[-\frac{d^2}{dx^2} + \frac{\lambda(\lambda+1)}{\sinh^2 x}\right]\psi_k(x) = k^2\psi_k(x)$. Due to the underlying $SU(1,1)$ structure of the scattering problem [12], $\psi_k(x)$ is proportional to $(\sinh x)^{\lambda+1}\,_2F_1((\lambda+1+ik)/2, (\lambda+1-ik)/2, \lambda+3/2; -\sinh^2 x)$, where $_2F_1$ is the hypergeometric function. The asymptotic states for $x \to \infty$ is

$$\psi_k(x) \to C\left(e^{ikx}\frac{\Gamma(ik)\Gamma(2\lambda+2)}{\Gamma(\lambda+1+ik)\Gamma(\lambda+1)} + e^{-ikx}\frac{\Gamma(-ik)\Gamma(2\lambda+2)}{\Gamma(\lambda+1-ik)\Gamma(\lambda+1)}\right) \tag{2}$$

the two-body scattering matrices are

$$S^+(k) = \frac{\Gamma(ik)\Gamma(1+\lambda-ik)}{\Gamma(-ik)\Gamma(1+\lambda+ik)}, \qquad S^-(k) = \frac{\Gamma(ik)\Gamma(\lambda-ik)}{\Gamma(-ik)\Gamma(\lambda+ik)}, \tag{3}$$

for $P_{ij} = \pm 1$, respectively [13].

Now returning to $|\sigma_i \sigma_j\rangle$ basis, it is straightforward to obtain particle-particle scattering amplitudes,

$$S^{\sigma_j \sigma_i}_{\sigma_i \sigma_j} \equiv u_1 = \frac{1}{2}(S^+ + S^-), \qquad S^{\sigma_i \sigma_j}_{\sigma_i \sigma_j} \equiv u_2 = \frac{1}{2}(S^+ - S^-), \tag{4}$$

for $\sigma_i \neq \sigma_j$. Note that for $\sigma_i = \sigma_j$, $S^{\sigma_i \sigma_i}_{\sigma_i \sigma_i} = u_1 + u_2$. Similarly, the scattering amplitudes of particles and antiparticles of different colors, $\sigma_i \neq \bar{\sigma}_j$ become

$$S^{\sigma_i \bar{\sigma}_j}_{\sigma_i \bar{\sigma}_j} \equiv r_1 = \frac{1}{2}(S^+ + S^-), \qquad S^{\bar{\sigma}_j \sigma_i}_{\sigma_i \bar{\sigma}_j} \equiv t_1 = \frac{1}{2}(S^+ - S^-), \tag{5}$$

where $\bar{\sigma}_j$ stands for the color of an antiparticle. (See Fig.1 for schematic definitions of the amplitudes.)
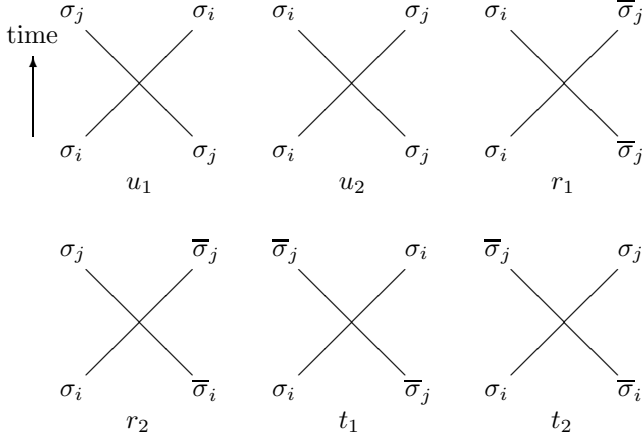

Fig.1: Scattering amplitudes of particles and antiparticles with colors

The scattering amplitudes of the same color have to be dealt with some care. The most general eigenstates of P_{ij} are now of the form $|\pm\rangle = \frac{1}{\sqrt{2}}(|A\rangle \pm |B\rangle)$, where $|A\rangle = \mathcal{N} \sum_{i=1}^{N} a_i |\sigma_i \bar{\sigma}_i\rangle$ and $|B\rangle = \mathcal{N} \sum_{i=1}^{N} a_i |\bar{\sigma}_i \sigma_i\rangle$ for some coefficients a_i's and $\mathcal{N} = (\sum_{i=1}^{N} a_i^2)^{-1/2}$. The scattering amplitudes for $|\pm\rangle$ are $S^\pm$ as before and if we express these in the $|\sigma_i \bar{\sigma}_i\rangle$ basis we find

$$\frac{1}{2}(S^+ + S^-) = r_1 + M r_2, \qquad \frac{1}{2}(S^+ - S^-) = t_1 + M t_2, \qquad \text{where} \qquad M = \mathcal{N}^2 \sum_{i,j}^{N} a_i a_j. \tag{6}$$

Comparing Eq.(5) with (6), we find $r_2 = t_2 = 0$. With $S^\pm$ given in Eq.(3), we obtain the scattering amplitudes of the $SU(N)$-invariant CSM as follows:

$$u_1 = t_1 = \frac{-ik}{\lambda + ik} \cdot \frac{\Gamma(ik)\Gamma(\lambda - ik)}{\Gamma(-ik)\Gamma(\lambda + ik)}, \quad u_2 = r_1 = \frac{\lambda}{\lambda + ik} \cdot \frac{\Gamma(ik)\Gamma(\lambda - ik)}{\Gamma(-ik)\Gamma(\lambda + ik)}, \quad r_2 = t_2 = 0. \tag{7}$$

Since we have derived the scattering amplitudes of the CSM which is integrable, it is reasonable to study the problem in the context of the factorizable scattering theory where the integrability requires the Yang-Baxter equations. As a candidate, we consider the Yang-Baxter equations with $SU(N)$-invariance because the particles and antiparticles of the CSM carry the $SU(N)$-color quantum numbers. For the nonrelativistic system where CPT-invariance does not hold anymore, one needs not implement the crossing symmetry.

The Yang-Baxter equations and unitarity condition give 15 equations for the amplitudes which are the same as the relativistic case, Berg et. al. [9]. Seven of these include neither r_2 nor t_2;

$$u_1(\theta)u_1(-\theta) + u_2(\theta)u_2(-\theta) = 1, \tag{8}$$
$$u_1(\theta)u_2(-\theta) + u_2(\theta)u_1(-\theta) = 0,$$
$$t_1(\theta)t_1(-\theta) + r_1(\theta)r_1(-\theta) = 1,$$
$$t_1(\theta)r_1(-\theta) + r_1(\theta)t_1(-\theta) = 0,$$

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$$u_1(\theta)t_1(\theta + \theta')r_1(\theta') = t_1(\theta)u_1(\theta + \theta')r_1(\theta'),$$
$$u_2(\theta)t_1(\theta + \theta')r_1(\theta') = r_1(\theta)r_1(\theta + \theta')t_1(\theta') + t_1(\theta)u_2(\theta + \theta')r_1(\theta'),$$
$$u_2(\theta)u_1(\theta + \theta')u_2(\theta') = u_1(\theta)u_2(\theta + \theta')u_2(\theta') + u_2(\theta)u_2(\theta + \theta')u_1(\theta'),$$

where θ is the spectral parameter. We see that this has the "minimal" solution of

$$u_1(\theta) = t_1(\theta) = \frac{-\theta}{\gamma + \theta} \cdot \frac{\Gamma(\theta)\Gamma(\gamma - \theta)}{\Gamma(-\theta)\Gamma(\gamma + \theta)}, \qquad u_2(\theta) = r_1(\theta) = \frac{\gamma}{\gamma + \theta} \cdot \frac{\Gamma(\theta)\Gamma(\gamma - \theta)}{\Gamma(-\theta)\Gamma(\gamma + \theta)}, \tag{9}$$

where γ is the arbitrary parameter. The rest of the equations are trivially satisfied with $r_2 = t_2 = 0$. Replacing θ with ik and interpreting the parameter γ as the coefficient of the hyperbolic interaction λ, these scattering amplitudes are identical to those of the multicomponent CSM.

We compute all other possible solutions which are listed in Table 1. It turns out that the classes I-VI are exactly the nonrelativistic limits of the six classes of solutions that Berg et. al. [9], i.e., imposing the crossing symmetry of $u_1(\theta) = t_1(i\pi - \theta)$, $u_2(\theta) = t_2(i\pi - \theta)$, and $r_1(\theta) = r_2(i\pi - \theta)$, which fixes the parameter γ to a certain value for each classes. The class VII corresponds to the multicomponent CSM. The rest are new solutions which do not have relativistic analogs.

In Table 1, the functions $U(\theta), R(\theta)$, and $T(\theta)$ satisfy the unitarity condition $U(\theta)U(-\theta) = 1$ etc., which can fix these functions with the minimality condition. In the class X, functions t_1, r_1 satisfy $t_1(\theta)t_1(-\theta) + r_1(\theta)r_1(-\theta) = 1$.

In our model, we have considered the case where the potentials between two particles irrespective of the species are all $1/\sinh^2 x$-type and shown that this corresponds to $SU(N)$-invariant factorizable scattering theory. We want to point out that there exists a more general exactly solvable potential (Pöschl-Teller) [12] which contains both $1/\sinh^2 x$ and $1/\cosh^2 x$ [15]. In ref. [7], it has been shown that the $O(2)$-invariant scattering theory (the sine-Gordon model) is related to the nonrelativistic Hamiltonian system where the $1/\sinh^2 x$ potential is for (anti)particle-(anti)particle scattering and the $1/\cosh^2 x$ for particle-antiparticle. It would be interesting to consider the case where particle-antiparticle scattering potential is different from particle-particle potential where particles carry color charges. We would like to emphasize the approach to the multicomponent CSM and the generalized Haldane-Shastry model [3,4,16] based on the factorizable S-matrix theory can be fruitful. We hope our approach can be generalized to other integrable Hamiltonian systems.

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Class	$u_1(\theta)$	$u_2(\theta)$	$r_1(\theta)$	$r_2(\theta)$	$t_1(\theta)$	$t_2(\theta)$
I	$\frac{\theta}{\gamma\pm\theta}U(\theta)$	$\frac{\gamma}{\theta}u_1(\theta)$	0	0	$T(\theta)$	0
II	$\frac{\theta}{\gamma\pm\theta}U(\theta)$	$\frac{\gamma}{\theta}u_1(\theta)$	0	0	$T(\theta)$	$-\frac{\gamma}{\frac{\gamma N}{2}+\theta}t_1(\theta)$
III	$t_1(\theta)$	$r_1(\theta)$	$\frac{\gamma}{\theta}t_1(\theta)$	$\frac{\gamma}{\gamma(1-N)-\theta}t_1(\theta)$	$\frac{\theta}{\gamma+\theta}U(\theta)$	$r_2(\theta)$
IV	$-t_1(\theta)$	$r_1(\theta)$	$-\frac{\gamma}{\theta}t_1(\theta)$	$\frac{\gamma}{-\gamma(1-N)-\theta}t_1(\theta)$	$\frac{\theta}{\gamma-\theta}U(\theta)$	$r_2(\theta)$
V	0	$r_1(\theta)$	$R(\theta)$	$r_1(\theta)$	0	$r_1(\theta)$
VI	0	$e^{\gamma\theta}r_1(\theta)$	$R(\theta)$	$-\frac{N(e^{2\gamma\theta}-1)}{N^2e^{2\gamma\theta}-1}r_1(\theta)$	0	$N^{-1}e^{-\gamma\theta}r_2(\theta)$
VII	$t_1(\theta)$	$r_1(\theta)$	$-\frac{\gamma}{\theta}t_1(\theta)$	0	$\frac{-\theta}{\gamma+\theta}U(\theta)$	0
VIII	$\frac{\theta}{\gamma\pm\theta}U(\theta)$	$\frac{\gamma}{\theta}u_1(\theta)$	$R(\theta)$	0	0	0
IX	0	$U(\theta)$	$R(\theta)$	0	0	$\gamma\left[U(\theta)-U(-\theta)\frac{R(\theta)}{R(-\theta)}\right]$
X	0	$U(\theta)$	$r_1(\theta)$	0	$\gamma\left[U(\theta)-U(-\theta)\frac{r_1(\theta)}{r_1(-\theta)}\right]$	0 or $-\frac{N}{2}t_1(\theta)$

Table 1: Complete solutions of nonrelativistic $SU(N)$ invariant Yang-Baxter equations